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Unsteady viscous flow in a pipe of slowly varying cross-section

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The steady streaming generated in a pipe of slowly varying cross-section when a purely oscillatory pressure difference is maintained between its ends is considered. It is assumed that the perturbation of the pipe wall in the r, θ plane is small compared with the characteristic thickness of the Stokes layer associated with the oscillatory motion of the fluid. The first-order steady streaming is evaluated for the cases when this characteristic thickness is large and small compared with a typical radius of the pipe. In both these limits it is found that the geometry of the pipe is crucial in determining the nature of the induced steady streaming. If the ends of the pipe have the same mean radius it is found that the steady streaming consists of regions of recirculation between the nodes of the pipe. Otherwise the steady streaming is of a larger order of magnitude and has a component which represents a net flow towards the wider end of the pipe.

1. Introduction

When a purely oscillatory viscous flow is set up over a curved surface the Reynolds stresses associated with the oscillatory motion of the fluid in the Stokes layer at the surface generate a steady component of velocity which persists away from the layer because of the action of viscosity. Several authors have discussed this type of flow adjacent to a circular cylinder (see for example Schlichting 1932; Riley 1965; Stuart 1966). Similar mathematical and physical ideas arise in the study of water waves (see Longuet-Higgins 1953). More recently Lyne (1971*b*) has used the method of conformal transformation to investigate the steady streaming induced by an oscillatory viscous flow adjacent to a wavy wall.

We shall use what is often called 'lubrication theory' to obtain the first-order steady streaming generated by a purely oscillatory pressure difference maintained between the ends of a pipe of slowly varying cross-section. Essentially the method requires that as well as the pipe cross-section varying slowly a modified Reynolds number associated with the flow be small. Manton (1971) has considered the corresponding steady problem by a similar method. Ramachandra Rao & Devanathan (1973) have recently considered pulsatile flow in pipes of slowly varying cross-section. We postpone a discussion of their work until the end of this section.

Suppose that the pipe is defined in cylindrical polar co-ordinates (r, θ, z) by

$$0 \leqslant r \leqslant D_0\{R(z/L) + \epsilon S(z/L) \cos M\theta\}, \quad 0 \leqslant z \leqslant KL, \qquad (1.1\,a, b)$$

where M clearly must be an integer, which we assume to be positive. This corresponds to a pipe which, for a given value of z, has its maximum radius at the points where $\cos M\theta = 1$ and minimum radius at the points where $\cos M\theta = -1$ if $S(z/L)$ is positive and vice versa if it is negative. We consider only pipes which are slightly non-axisymmetric in the sense that ϵ is assumed small in the following work. It is clear that pipes of more general shape can, as long as they are only slightly non-axisymmetric, be treated by Fourier analysing the θ dependence of the pipe radius. It is of interest to notice that the case $M = 2$ corresponds to a pipe with an elliptical cross-section of small eccentricity. The major and minor axes at any value of z are given by the lines $\theta = 0$, $\frac{1}{2}\pi$ respectively if $S(z/L)$ is positive and vice versa if it is negative.

We define a parameter δ representing the order of magnitude of the rate of change of the pipe radius by

$$\delta = D_0^2/L^2, \qquad (1.2)$$

where D_0 and L are characteristic lengths in the r and z directions respectively. We assume throughout that

$$\delta \ll 1. \qquad (1.3)$$

If ω is the frequency of the applied pressure difference and ν is the kinematic viscosity we define a frequency parameter σ by

$$\sigma = \omega D_0^2/\nu. \qquad (1.4)$$

Thus $\sigma^{-\frac{1}{2}}$ represents the ratio of the characteristic thickness of the Stokes layer formed by the oscillatory motion of the fluid to the typical radius of the pipe. We assume that ϵ is such that

$$\epsilon\sigma^{\frac{1}{2}} \ll 1 \qquad (1.5)$$

so that the perturbation of the pipe wall in the r, θ plane is small compared with the characteristic thickness of the Stokes layer.

If U_0 is a typical axial velocity, brought about by the driving pressure difference, then we define a modified Reynolds number R_M by

$$R_M = U_0 D_0^2/L\nu. \qquad (1.6)$$

R_M is taken to be small throughout this work. It is clear that U_0 will be proportional to the amplitude of the applied pressure difference. The procedure adopted below is as follows.

In §2 we formulate the non-dimensional equations governing the flow and determine the so-called 'Stokes flow' by putting the parameters R_M and δ equal to zero. The steady streaming first appears in the order-R_M correction to the Stokes flow and this is evaluated in the high and low frequency limits in §§3 and 4 respectively. In §5 we give a brief discussion of the results obtained in the previous two sections and the relevance of these results to some physiological flows.

Having given a brief description of the work presented in this paper we now discuss the work of Ramachandra Rao & Devanathan (1973). These authors considered pulsatile flow in pipes of slowly varying radius. They assumed that the ratio k of the orders of magnitude of typical oscillatory and steady axial velocities was small. They then obtained a solution by expanding the stream function in powers of k and parameters corresponding to R_M and δ. (The parameters corresponding to R_M and δ are in fact $R_e\epsilon$ and ϵ^2 in their notation.) The terms in the expansion evaluated by them were those of order 1, R_M, k and $R_M k$. The first two of these terms are clearly steady and are identical to those evaluated by Manton (1971) and Hall (1973). The order-k term in the expansion of Ramachandra Rao & Devanathan is oscillatory and can be shown to be the order-one term of our expansion. The remaining term evaluated by them is another oscillatory term and is produced by the nonlinear interaction of the order-R_M and order-k terms. The steady streaming terms would first appear at order $R_M k^2$ in their expansion and, as stated earlier, were not evaluated by them. It should be said that the condition that k be small is not required in order to solve their problem. It can be shown that the work presented in this paper can be easily combined with the work on the steady problem given in the author's thesis (Hall 1973) to obtain a solution to the pulsatile flow problem.

2. Equations of motion and the Stokes flow

We consider viscous incompressible flow in a pipe defined in cylindrical polar co-ordinates by (1.1). We define p, ν, ρ and t to be the pressure, kinematic viscosity, density and time respectively. We assume that the pressure difference between the ends is given by

$$p(R(K)+\epsilon S(K)\cos M\theta,\theta,KL,t)-p(R(0)+\epsilon S(0)\cos M\theta,\theta,0,t) = C_0\sin\omega t. \quad (2.1)$$

We shall in fact see that to the order in R_M and δ to which we work p is independent of r and so it is not necessary to specify the pressure in (2.1) evaluated at the pipe wall. However at higher order this is not the case and p is a function of r, thus requiring the boundary condition to be specified at the pipe wall. We define dimensionless variables τ, η and ζ by

$$\tau = \omega t, \quad \eta = r/D_0, \quad \zeta = z/L. \quad (2.2a, b, c)$$

If (u,v,w) is the velocity vector corresponding to (r,θ,z) we define a dimensionless velocity by

$$(g,h,f) = (\delta^{-\frac{1}{2}}/U_0)(u,v,\delta^{\frac{1}{2}}w), \quad (2.3)$$

where U_0 is again a typical velocity along the pipe. We define a dimensionless pressure p^+ by

$$p^+ = pD_0^2/\rho\nu LU_0. \quad (2.4)$$

The momentum and continuity equations can then be written in the form

$$\sigma\delta\frac{\partial g}{\partial\tau}+R_M\delta\left\{g\frac{\partial g}{\partial\eta}+\frac{h}{\eta}\frac{\partial g}{\partial\theta}+f\frac{\partial g}{\partial\zeta}-\frac{h^2}{\eta}\right\} = -\frac{\partial p^+}{\partial\eta}+\delta\{\nabla^2-\eta^{-2}\}g-\frac{2\delta}{\eta^2}\frac{\partial h}{\partial\theta}+\delta^2\frac{\partial^2 g}{\partial\zeta^2}, \quad (2.5a)$$

$$\sigma\delta\frac{\partial h}{\partial \tau}+R_M\delta\left\{g\frac{\partial h}{\partial \eta}+\frac{h}{\eta}\frac{\partial h}{\partial \theta}+f\frac{\partial h}{\partial \zeta}+\frac{gh}{\eta}\right\}=-\frac{1}{\eta}\frac{\partial p^+}{\partial \theta}+\delta\{\nabla^2-\eta^{-2}\}h+\frac{2\delta}{\eta^2}\frac{\partial g}{\partial \theta}+\delta^2\frac{\partial^2 h}{\partial \zeta^2}, \tag{2.5b}$$

$$\sigma\frac{\partial f}{\partial \tau}+R_M\left\{g\frac{\partial f}{\partial \eta}+\frac{h}{\eta}\frac{\partial f}{\partial \theta}+f\frac{\partial f}{\partial \zeta}\right\}=-\frac{\partial p^+}{\partial \zeta}+\nabla^2 f+\delta\frac{\partial^2 f}{\partial \zeta^2}, \tag{2.5c}$$

$$\frac{\partial}{\partial \eta}(\eta g)+\frac{\partial h}{\partial \theta}+\eta\frac{\partial f}{\partial \zeta}=0, \tag{2.5d}$$

where $$\nabla^2\equiv\partial^2/\partial\eta^2+\eta^{-1}\partial/\partial\eta+\eta^{-2}\partial^2/\partial\theta^2$$

and δ, σ and R_M are as defined by (1.2), (1.4) and (1.6) respectively. These equations must be solved subject to there being no relative velocity at the pipe wall. Thus we require that

$$g=h=f=0 \quad \text{at} \quad \eta=R(\zeta)+\epsilon S(\zeta)\cos M\theta, \tag{2.6a, b, c}$$

and from (2.1) it follows that

$$p^+(R+\epsilon S\cos M\theta,\theta,K,\tau)-p^+(R+\epsilon S\cos M\theta,\theta,0,\tau)=\alpha\sin\tau, \tag{2.7}$$

where α is defined by

$$\alpha=C_0D_0^2/\rho\nu LU_0. \tag{2.8}$$

The remaining conditions required to specify the problem completely are kinematical in origin. We require that, at $\eta=0$, p^+ and f must be independent of θ, whilst g and h must vary like $\cos\theta$ and $\sin\theta$ there. A helpful reference where these conditions are discussed in more detail is Batchelor & Gill (1962).

We seek a solution of the system just specified by expanding g, f, h and p^+ in the form

$$g=G_{00}+\epsilon G_{01}+R_MG_{10}+\epsilon R_MG_{11}+O(\delta R_{M1}^2,\epsilon^2),\text{ etc.} \tag{2.9}$$

The Stokes flow is then obtained by putting the parameters R_M and δ equal to zero everywhere. We now solve for this flow up to and including terms of order ϵ. We first write

$$G_{00}=\tfrac{1}{2}\{g_{00}e^{i\tau}+\tilde{g}_{00}e^{-i\tau}\},\text{ etc.,} \tag{2.10}$$

where a tilde denotes a complex conjugate and the functions f_{00}, g_{00}, h_{00} and p_{00} are all independent of τ. If we substitute for G_{00}, etc., from (2.10) into (2.9) and then substitute the resulting expressions into (2.5) and equate terms of order unity which are proportional to $e^{i\tau}$ we obtain

$$\partial p_{00}/\partial\eta=\partial p_{00}/\partial\theta=0, \tag{2.11a, b}$$

$$\partial p_{00}/\partial\zeta=\{\nabla^2-i\sigma\}f_{00}, \tag{2.11c}$$

$$\frac{\partial}{\partial \eta}(\eta g_{00})+\frac{\partial h_{00}}{\partial \theta}+\eta\frac{\partial f_{00}}{\partial \zeta}=0. \tag{2.11d}$$

Thus we see that p_{00} is independent of r and θ, and we can use (2.11c) to show that if solutions of Bessel's equation which are singular at $\eta=0$ are rejected then f_{00} is given by

$$f_{00}=-p_{00}'/i\sigma+A_{01}J_0(s\eta), \tag{2.12}$$

where we have rejected all θ-dependent solutions. The vanishing of these terms would otherwise be obtained by applying the boundary conditions. From now on a prime (except on a Bessel function) denotes a derivative with respect to ζ, and s is defined by

$$s = (-i\sigma)^{\frac{1}{2}}. \tag{2.13}$$

The coefficient A_{01} appearing in (2.12) is a function of ζ and will be determined later. If we now substitute for f_{00} from (2.12) into (2.11d) we can show that

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta g_{00}) + \frac{1}{\eta}\frac{\partial h_{00}}{\partial\theta} = \frac{p''_{00}}{i\sigma} - A'_{01}J_0(s\eta). \tag{2.14}$$

In order to solve for g_{00} and h_{00} we need another equation linking these quantities. This is found by eliminating the pressure from (2.5a, b) and then equating terms of order δ after substituting for g, h and f from (2.9). If we then again define G_{00} and H_{00} as in (2.10) we obtain

$$\{\nabla^2 - i\sigma\}\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta h_{00}) - \frac{1}{\eta}\frac{\partial g_{00}}{\partial\theta}\right] = 0,$$

which we solve to get
$$\frac{\partial}{\partial\eta}(\eta h_{00}) - \frac{\partial g_{00}}{\partial\theta} = B_{01}\eta J_0(s\eta), \tag{2.15}$$

where we have again rejected any θ-dependent solutions. The vanishing of these solutions would otherwise be obtained on applying the boundary conditions later. The coefficient B_{01} is a function of ζ and will be determined later. The solutions of Bessel's equation which lead to terms in g_{00} and h_{00} singular at $\eta = 0$ have been rejected. We can eliminate g_{00} from (2.14) and (2.15) to get an equation for h_{00} whose solution which is regular at $\eta = 0$ is

$$h_{00} = (i\sigma)^{-1}B_{01}\,dJ_0(s\eta)/d\eta. \tag{2.16}$$

We can now substitute for h_{00} from (2.16) into (2.14) and after multiplying by η we can integrate from 0 to η to get

$$g_{00} = [p''_{00}\eta - 2A'_{01}\,dJ_0(s\eta)/d\eta]/2i\sigma, \tag{2.17}$$

where we have used the fact that g_{00} is regular at $\eta = 0$ to show that ηg_{00} is zero there. We can repeat the above calculations to show that the order-ϵ terms in the Stokes flow are given by

$$\begin{aligned} g_{01} = {} & -\frac{1}{i\sigma}\{a'_{M1}\cos M\theta + a'_{M2}\sin M\theta\}\frac{dJ_M}{d\eta}(s\eta) \\ & + \frac{M}{i\sigma}\{b_{M1}\sin M\theta - b_{M2}\cos M\theta\}\frac{J_M(s\eta)}{\eta} \\ & + \{c_{M1}\sin M\theta - c_{M2}\cos M\theta\}\eta^{M-1} + p''_{01}\eta/2i\sigma, \end{aligned} \tag{2.18a}$$

$$\begin{aligned} h_{01} = {} & \frac{M}{i\sigma}\{a'_{M1}\sin M\theta - a'_{M2}\cos M\theta\}\frac{J_M(s\eta)}{\eta} \\ & + \frac{1}{i\sigma}\{b_{M1}\cos M\theta + b_{M2}\sin M\theta\}\frac{dJ_M(s\eta)}{d\eta} \\ & + \{c_{M1}\cos M\theta + c_{M2}\sin M\theta\}\eta^{M-1} + \frac{b_{01}}{i\sigma}\frac{dJ_0(s\eta)}{d\eta}, \end{aligned} \tag{2.18b}$$

$$f_{01} = -p'_{01}/i\sigma + \{a_{M1}\cos M\theta + a_{M2}\sin M\theta\}J_M(s\eta), \tag{2.18c}$$

where a_{M1}, etc., are functions of ζ to be determined later and g_{01}, etc., are defined by

$$G_{01} = \tfrac{1}{2}\{g_{01}e^{i\tau} + \tilde{g}_{01}e^{-i\tau}\}, \text{ etc.} \tag{2.19}$$

For convenience we have rejected any θ-dependent solutions other than those proportional to $\cos M\theta$ or $\sin M\theta$. The vanishing of these solutions would otherwise be obtained on applying the boundary conditions. In order to solve for the unknown functions of ζ in f_{00}, etc., we must consider the boundary conditions on the velocity and the pressure. From (2.6), (2.7), (2.9), (2.10) and (2.19) it follows that these may be written in the form

$$g_{00} + \epsilon g_{01} = h_{00} + \epsilon h_{01} = f_{00} + \epsilon f_{01} = O(\epsilon^2) \quad \text{at} \quad \eta = R + \epsilon S \cos M\theta, \tag{2.20a, b, c}$$

$$p_{00}(K) - p_{00}(0) = -i\alpha, \quad p_{01}(K) - p_{01}(0) = 0. \tag{2.20d, e}$$

We can use (2.12) and (2.18c) to show that the condition (2.20c) gives

$$O(\epsilon^2) = -\frac{1}{i\sigma}\{p'_{00} + \epsilon p'_{01}\} + \{\epsilon a_{M1}\cos M\theta + \epsilon a_{M2}\sin M\theta\}$$
$$\times\{J_M(sR) + \epsilon sS\cos(M\theta)\, J'_M(sR)\} + A_{01}\{J_0(sR) + \epsilon sSJ'_0(sR)\},$$

where from now on a prime on a Bessel function denotes the derivative with respect to its argument, and we have replaced $J_M(sR + \epsilon sS\cos M\theta)$ by its Taylor series expansion about sR. The validity of this expansion is ensured by (1.5). The coefficients A_{01}, a_{M1} etc., are then found by equating terms proportional to $1, \epsilon\cos M\theta$, etc. If we then substitute the values of the coefficients obtained by this process into (2.12) and (2.18c), we can write f_{00} and f_{01} in the form

$$f_{00} = -\frac{p'_{00}}{i\sigma}\left\{1 - \frac{J_0(s\eta)}{J_0(sR)}\right\}, \tag{2.21a}$$

$$f_{01} = -\frac{p'_{01}}{i\sigma}\left\{1 - \frac{J_0(s\eta)}{J_0(sR)}\right\} - \frac{sSp'_{00}J'_0(sR)\, J_M(s\eta)\cos M\theta}{i\sigma J_0(sR)\, J_M(sR)}. \tag{2.21b}$$

We can similarly substitute the expressions (2.16), (2.17) and (2.18a, b) into (2.20a, b) and expand in powers of ϵ. If we then equate terms independent of ϵ and θ in (2.20a) we obtain

$$\left\{\frac{R^2}{2} - \frac{RJ_1(sR)}{sJ_0(sR)}\right\}p''_{00} - RR'\left\{\frac{J_1(sR)}{J_0(sR)}\right\}^2 p'_{00} = 0, \tag{2.22}$$

which is the Reynolds equation for the pressure and can be integrated to give

$$p'_{00} = \frac{E}{\{\frac{1}{2}R^2 - RJ_1(sR)/sJ_0(sR)\}}. \tag{2.23}$$

The constant E is obtained by integrating both sides of (2.23) from $\zeta = 0$ to $\zeta = K$ and using (2.20d). We thus obtain

$$E = -i\alpha \Bigg/ \int_0^K \frac{d\zeta}{\{\frac{1}{2}R^2 - RJ_1(sR)/sJ_0(sR)\}}. \tag{2.24}$$

If we equate terms proportional to ϵ and independent of θ in (2.20a) we obtain the Reynolds equation for p_{01}. If this equation is solved subject to (2.20e) we find that $p'_{01} = 0$. If we equate terms of order unity and independent of θ in (2.20*b*) we find that $B_{01} = 0$. Similarly we find by equating terms independent of θ of order ϵ in (2.20*b*) that $b_{01} = 0$. The coefficients b_{M1}, etc., are found by equating terms proportional to $\epsilon \cos M\theta$ and $\epsilon \sin M\theta$ in (2.20*a, b*). We can then show that

$$g_{00} = \frac{p''_{00}}{2i\sigma}\left\{\eta - \frac{RJ_1(s\eta)}{J_1(sR)}\right\}, \tag{2.25a}$$

$$g_{01} = \left[\left(\frac{\eta}{R}\right)^{M-1}\{sa'_{M1}J_{M-1}(sR)+K(\zeta)\} - a'_{M1}sJ_{M-1}(s\eta) + \frac{K(\zeta)\,MJ_M(s\eta)}{J'_M(sR)\,s\eta}\right]\frac{\cos M\theta}{i\sigma}, \tag{2.25b}$$

$$h_{00} = 0, \tag{2.25c}$$

$$h_{01} = \left[-\left(\frac{\eta}{R}\right)^{M-1}\{sa'_{M1}J_{M-1}(sR)+K(\zeta)\} + a'_{M1}sJ_{M-1}(s\eta) + \frac{K(\zeta)}{sJ'_M(sR)}\frac{dJ_M(s\eta)}{d\eta}\right]\frac{\sin M\theta}{i\sigma}, \tag{2.25d}$$

where
$$K(\zeta) = \frac{sA'_{01}S[J'_0(sR) - sRJ''_0(sR)]\,J'_M(sR)}{RJ_{M-1}(sR)}, \tag{2.26a}$$

$$A'_{01} = \frac{Rp''_{00}}{2sJ'_0(sR)}, \quad a_{M1} = \frac{sSJ_1(sR)\,p'_{00}}{i\sigma J_0(sR)\,J_M(sR)}. \tag{2.26b, c}$$

(For more details of the determination of the above expressions see Hall 1973.)

3. Calculation of the steady streaming for large σ

When σ is large the thickness of the Stokes layer associated with the oscillatory motion of the fluid is very small compared with a typical value of the pipe radius. We first discuss the nature of the Stokes flow for large σ. If we use the asymptotic expansion for Bessel functions of large argument in (2.23) and (2.24) we can show that

$$p'_{00} = -\frac{i\alpha}{R^2}\left\{\beta_0 + \frac{2\beta_0}{R(i\sigma)^{\frac{1}{2}}} - \frac{2\beta_1}{(i\sigma)^{\frac{1}{2}}} + O(\sigma^{-1})\right\}, \tag{3.1}$$

where
$$\beta_0 = \left\{\int_0^K \frac{d\zeta}{R^2}\right\}^{-1}, \quad \beta_1 = \beta_0^2\int_0^K \frac{d\zeta}{R^3}, \tag{3.2a, b}$$

and we now choose α such that $\alpha = \beta_0^{-1}$. This is equivalent to redefining the typical axial velocity U_0 in terms of the amplitude of the applied pressure difference. We can then write (3.1) in the form

$$p'_{00} = -\frac{i}{R^2}\left\{1 + \frac{2}{R(i\sigma)^{\frac{1}{2}}} - \frac{2\beta_1}{\beta_0(i\sigma)^{\frac{1}{2}}} + O(\sigma^{-1})\right\}. \tag{3.3}$$

If we define a Stokes-layer variable η' by

$$\eta' = (R-\eta)\left(\tfrac{1}{2}\sigma\right)^{\frac{1}{2}}, \tag{3.4}$$

then by expanding the Bessel functions in (2.21) and (2.25) for large $|s\eta|$ and $|sR|$ and using (3.3) we can show that in the Stokes layer

$$f_{00} \sim (\sigma R^2)^{-1}\{1 - \exp(-(1+i)\,\eta') + O(\sigma^{-\frac{1}{2}})\}, \tag{3.5a}$$

$$g_{00} \sim (R'/\sigma R^2)\{1 - \exp(-(1+i)\,\eta') + O(\sigma^{-\frac{1}{2}})\}, \tag{3.5b}$$

$$f_{01} \sim [S\exp(-(1+i)\eta')/(-i\sigma)^{\frac{1}{2}}R^2+O(\sigma^{-1})]\cos M\theta, \tag{3.5c}$$

$$g_{01} \sim [SR'\exp(-(1+i)\eta')\eta'/(-i\sigma)^{\frac{1}{2}}R^2+O(\sigma^{-1})]\cos M\theta, \tag{3.5d}$$

$$h_{01} \sim O(\sigma^{-1}). \tag{3.5e}$$

If we put $S = R' = 0$ in (3.5) we see that the flow in the Stokes layer reduces to the Stokes shear-wave solution for flow in a circular pipe. We can also see from (3.5) that the order-ϵ corrections to the axisymmetric flow have a dominant term which decays to zero at the edge of the Stokes layer. We shall refer to the region away from the Stokes layer as the 'outer' layer.

If we substitute for g, h, f and p^+ from (2.9) into (2.5) and equate terms of order R_M we obtain

$$\partial p_{10}/\partial\eta = \partial p_{10}/\partial\theta = 0, \tag{3.6a, b}$$

$$\left[\nabla^2-\sigma\frac{\partial}{\partial\tau}\right]F_{10} = \frac{\partial p_{10}}{\partial\zeta}+\left\{F_{00}\frac{\partial F_{00}}{\partial\zeta}+\frac{H_{00}}{\eta}\frac{\partial F_{00}}{\partial\theta}+G_{00}\frac{\partial F_{00}}{\partial\eta}\right\}, \tag{3.7a}$$

$$\frac{\partial}{\partial\eta}(\eta G_{10})+\frac{\partial H_{10}}{\partial\theta}+\eta\frac{\partial F_{10}}{\partial\zeta} = 0. \tag{3.7b}$$

We recall that, in §2, F_{00} and G_{00} represented the axisymmetric solution and F_{01}, G_{01} and H_{01} gave the order-ϵ non-axisymmetric correction to this solution. Similarly F_{10} and G_{10} will represent the order-R_M axisymmetric solution and F_{11}, G_{11} and H_{11} will give the order-ϵR_M non-axisymmetric correction to this solution. Thus we drop the θ dependence in (3.7) and put $H_{10} = 0$. The relevant boundary conditions for F_{10} and G_{10} are

$$F_{10} = G_{10} = 0 \quad \text{at} \quad \eta = R, \tag{3.8a, b}$$

and from (2.7) and (2.9) if follows that

$$P_{10}(K)-P_{10}(0) = 0. \tag{3.9}$$

We now solve the axisymmetric problem. If we substitute for F_{00}, G_{00} and P_{00} from (2.10) into (3.6) we can see that F_{10}, G_{10} and P_{10} will have both steady and unsteady components, the unsteady components being proportional to $\cos 2\tau$ or $\sin 2\tau$. Suppose that we denote the steady parts of F_{10}, G_{10} and P_{10} by f_s, g_s and p_s respectively, then we can use (2.10) and (3.6) to show that

$$\left\{\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right\}f_s = p_s'+\frac{1}{4}\left\{f_{00}\frac{\partial\tilde{f}_{00}}{\partial\zeta}+\tilde{f}_{00}\frac{\partial f_{00}}{\partial\zeta}+g_{00}\frac{\partial\tilde{f}_{00}}{\partial\eta}+\tilde{g}_{00}\frac{\partial f_{00}}{\partial\eta}\right\}, \tag{3.10a}$$

$$\partial(\eta g_s)/\partial\eta+\eta\,\partial f_s/\partial\zeta = 0. \tag{3.10b}$$

We shall obtain solutions of (3.10) in the Stokes and outer layers separately and then match the solutions where the layers meet. We denote f_s in the Stokes and outer layers by f_s^i and f_s^o respectively. We can show from (2.21), (2.25), (2.26), (3.3), (3.4) and (3.10) that f_s^i and f_s^o satisfy the equations

$$\left\{\frac{\partial^2}{\partial\eta'^2}-\frac{1}{[(\frac{1}{2}\sigma)^{\frac{1}{2}}R-\eta']}\frac{\partial}{\partial\eta'}\right\}f_s^i = \frac{2}{\sigma}\Phi(\zeta,\sigma)$$
$$+\frac{R'}{\sigma^3R^5}\{\eta'[\cos\eta'+\sin\eta']e^{-\eta'}+4\cos\eta' e^{-\eta'}-\sin\eta' e^{-\eta'}-2e^{-2\eta'}+O(\sigma^{-\frac{1}{2}})\}, \tag{3.11a}$$

$$\left\{\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right\}f_s^o = \Phi(\zeta,\sigma), \tag{3.11b}$$

where $$\Phi(\zeta,\sigma) = p_s' + (4\sigma^2)^{-1}\{\tilde{p}_{00}' p_{00}'' + p_{00}' \tilde{p}_{00}''\}, \tag{3.12}$$

and it follows from (3.8) that

$$f_s^i = g_s^i = 0 \quad \text{at} \quad \eta' = 0 \tag{3.13a, b}$$

and f_s^o and g_s^o must be regular at $\eta = 0$. The pressure p_s can be shown from (3.9) to satisfy the following condition:

$$p_s(K) - p_s(0) = 0. \tag{3.14}$$

We can write the solutions of (3.11) in the form

$$f_s^i = \frac{\Phi}{2\sigma}\{\eta'^2 - (2\sigma)^{\frac{1}{2}} R\eta'\} + A(\zeta,\sigma) + B(\zeta,\sigma)\log\left\{R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right\}$$
$$+ \frac{R'}{2\sigma^3 R^5}\{\eta'[\cos\eta' - \sin\eta']e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + O(\sigma^{-\frac{1}{2}})\}, \tag{3.15a}$$

$$f_s^o = \tfrac{1}{4}\Phi\{\eta^2 - R^2\} + C(\zeta,\sigma), \tag{3.15b}$$

where A, B and C are for the moment unknown functions of ζ and σ and we have rejected solutions of (3.11b) which are singular at $\eta = 0$. If f_s^i and f_s^o are to match at the edge of the Stokes layer we require that

$$A = C, \quad B = 0.$$

Then using (3.13a) we obtain

$$A = R'/\sigma^3 R^5 + O(\sigma^{-\frac{7}{2}}), \tag{3.16}$$

and so we can write f_s^i and f_s^o as follows:

$$f_s^i = \frac{\Phi}{2\sigma}\{\eta'^2 - (2\sigma)^{\frac{1}{2}} R\eta'\}$$
$$+ \frac{R'}{2\sigma^3 R^5}\{\eta'[\cos\eta' - \sin\eta']e^{-\eta'} - 6\sin\eta' e^{-\eta'} - \cos\eta' e^{-\eta'} - e^{-2\eta'} + 2 + O(\sigma^{-\frac{1}{2}})\}, \tag{3.17a}$$

$$f_s^o = \tfrac{1}{4}\Phi\{\eta^2 - R^2\} + R'/\sigma^3 R^5 + O(\sigma^{-\frac{7}{2}}). \tag{3.17b}$$

In order to find Φ, and hence p_s', we must calculate the radial velocity component. Suppose that we denote g_s in the Stokes and outer layers by g_s^i and g_s^o respectively. In the Stokes layer (3.10b) becomes

$$\frac{\partial}{\partial\eta'}\left\{\left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]g_s^i\right\} - \left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]\left\{R'\frac{\partial f_s^i}{\partial\eta'} + \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\frac{\partial f_s^i}{\partial\zeta}\right\} = 0 \tag{3.18a}$$

and in the outer layer we have

$$\frac{\partial}{\partial\eta}(\eta g_s^o) + \eta\frac{\partial f_s^o}{\partial\zeta} = 0. \tag{3.18b}$$

If we substitute for f_s^i from (3.17a) into (3.18a) and integrate from 0 to a point η' in the Stokes layer we have

$$\left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]g_s^i = -\frac{\Phi'}{16}\left\{\left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]^4 - 2R^2\left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]^2 + R^4\right\}$$
$$+\left\{\frac{\Phi R' R}{4} - \left(\frac{R'}{2\sigma^3 R^5}\right)'(1 + O(\sigma^{-\frac{1}{2}}))\right\}\left\{\left[R - \left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]^2 - R^2\right\}$$

$$+\frac{1}{\sigma^{\frac{1}{2}}}\left\{\frac{R'}{2^{\frac{1}{2}}R^4}\right\}'\left\{\eta'\sin\eta' e^{-\eta'}+4\cos\eta' e^{-\eta'}+3\sin\eta' e^{-\eta'}+\frac{e^{-2\eta'}}{2}-\frac{9}{2}\right\}$$

$$+\frac{R'^2}{2\sigma^3R^5}\left[R-\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\eta'\right]\{\eta'[\cos\eta'-\sin\eta']e^{-\eta'}-6\sin\eta' e^{-\eta'}-\cos\eta' e^{-\eta'}-e^{-2\eta'}\}$$

$$+\frac{R'^2}{\phi^3R^4}+O(\sigma^{-\frac{7}{2}}), \qquad (3.19a)$$

where we have used (3.13b) to show that $[R-(2/\sigma)^{\frac{1}{2}}\eta']g_s^i$ is zero at $\eta'=0$. If we substitute for f_s^o from (3.17b) into (3.18b) and integrate from 0 to a point η still in the outer layer we obtain

$$\eta g_s^o = -\tfrac{1}{16}\Phi'\{\eta^4-2\eta^2R^2\}+\{\tfrac{1}{4}\Phi R'R-(R'/2\sigma^3R^5)'+O(\sigma^{-\frac{7}{2}})\}\eta^2, \quad (3.19b)$$

where we have used the fact that g_s^o is regular at $\eta=0$ to show that ηg_s^o is zero there. We now explain why we have evaluated only some of the terms of order $\sigma^{-\frac{7}{2}}$ in (3.19a, b). The terms of this order which are given explicitly are those which arise from the order-σ^{-3} terms in f_s^i and f_s^o through the equation of continuity. However, terms of similar order will arise from the order-$\sigma^{-\frac{7}{2}}$ terms in f_s^i and f_s^o, again through the equation of continuity, and these are the terms which are not given explicitly. The essential physical difference between the terms is that the latter terms in g_s^i, when combined with the order $\sigma^{-\frac{7}{2}}$ terms in f_s^i, give a resultant velocity parallel to the pipe wall, whilst the other terms lead to a component of velocity normal to the pipe wall. We shall in fact see that in the evaluation of the stream functions in the Stokes layer up to order $\sigma^{-\frac{7}{2}}$ the terms not shown explicitly are not required. If we use the condition that (3.19a, b) must match at the edge of the Stokes layer we obtain

$$0=-\frac{\Phi'R^4}{16}-\frac{\Phi R^3R'}{4}+\left(\frac{R'}{2\sigma^3R^5}\right)'R^2+\frac{R'^2}{\sigma^3R^4}-\frac{9}{2^{\frac{1}{2}}\sigma^{\frac{7}{2}}}\left(\frac{R'}{R^4}\right)'+O(\sigma^{-\frac{7}{2}}), \quad (3.20)$$

which we integrate once to get

$$Q=\frac{\Phi R^4}{16}-\left(\frac{R'}{2\sigma^3R^3}\right)\left[1-\frac{9}{(2\sigma)^{\frac{1}{2}}R}\right]+O(\sigma^{-\frac{7}{2}}), \qquad (3.21)$$

where Q is an unknown constant which we can determine by substituting for Φ from (3.12) into (3.21) and replacing p'_{00} by its asymptotic form (3.3). If we then integrate from $\zeta=0$ to $\zeta=K$ and use (3.14) we find that

$$Q=\left\{\frac{R^{-4}(K)-R^{-4}(0)}{64}+O(\sigma^{-\frac{1}{2}})\right\}\Big/\sigma^2\int_0^K\frac{d\zeta}{R^4}. \qquad (3.22)$$

Thus if the ends of the pipe have the same mean radius the terms of order σ^{-2} in (3.22) vanish. It can in fact be shown that all higher-order terms also vanish in this case. When Q is not zero, if the dominant outer velocity, which is of course given by the terms proportional to Q, is expressed in inner variables it is then identical to the inner-expansion terms proportional to Q, so that as far as the dominant term in the expansion of the velocity is concerned there is no need to distinguish between the two regions. If we consider Φ and Q as given by (3.21) and (3.22) we can use (3.17) and (3.19) to show that f_s and g_s can be written in the form

$$f_s = A_0[(\eta^2 - R^2)/\sigma^2 R^4 + O(\sigma^{-\frac{5}{2}})], \tag{3.23a}$$

$$g_s = A_0[R'\eta(\eta^2 - R^2)/\sigma^2 R^5 + O(\sigma^{-\frac{5}{2}})], \tag{3.23b}$$

where

$$A_0 = \{R^{-4}(K) - R^{-4}(0)\}\Big/ 16\int_0^K \frac{d\rho}{R^4}. \tag{3.23c}$$

If we introduce a stream function ψ_s defined by

$$\eta f_s = \partial\psi_s/\partial\eta, \quad \eta g_s = -\partial\psi_s/\partial\zeta \tag{3.24a, b}$$

then we can show that

$$\psi_s = \frac{A_0}{4\sigma^2}\left\{\frac{\eta^4 - 2\eta^2 R^2}{R^4}\right\} + O(\sigma^{-\frac{5}{2}}), \tag{3.25}$$

and this represents a steady flow towards the wider end of the pipe.

When Q is zero we can use (3.20) and (3.21) to show that f_s^i and g_s^i, given by (3.17a) and (3.19a), may be written in the form

$$f_s^i = \frac{R'}{2\sigma^3 R^5}\{\eta'[\cos\eta' - \sin\eta']\,e^{-\eta'} - 6\sin\eta'\,e^{-\eta'} - \cos\eta'\,e^{-\eta'} - e^{2\eta'} + 2 + O(\sigma^{-\frac{1}{2}})\}, \tag{3.26a}$$

$$g_s^i = R' f_s^i + \frac{1}{R}\left(\frac{R'}{2^{\frac{1}{2}}\sigma^{\frac{7}{2}}R^4}\right)'\left\{\eta'\sin\eta'\,e^{-\eta'} + 4\cos\eta'\,e^{-\eta'} + 3\sin\eta'\,e^{-\eta'} + \frac{e^{-2\eta'}}{2} + 4\eta' - \tfrac{9}{2}\right\} + O(\sigma^{-\frac{7}{2}}). \tag{3.26b}$$

We see from (3.26) that the order-σ^{-3} term in (3.26b) is just R' times that in (3.26a). Thus at any point in Stokes layer the dominant velocity is parallel to the pipe wall. We can also show that the order-$\sigma^{-\frac{7}{2}}$ terms not shown explicitly in (3.26) similarly represent a velocity parallel to the pipe wall. We define a stream function ψ_s^i in the Stokes layer by

$$f_s^i = \frac{-(\frac{1}{2}\sigma)^{\frac{1}{2}}}{(R - (2/\sigma)^{\frac{1}{2}}\eta')}\frac{\partial\psi_s^i}{\partial\eta'}, \quad g_s^i = \frac{-1}{(R - (2/\sigma)^{\frac{1}{2}}\eta')}\frac{\partial\psi_s^i}{\partial\zeta} + R' f_s^i. \tag{3.27a, b}$$

We can then use (3.26) to show that

$$\psi_s^i = (-R'/2^{\frac{3}{2}}\sigma^{\frac{7}{2}}R^4)\{2\eta'\sin\eta'\,e^{-\eta'} + 8\cos\eta'\,e^{-\eta'} + 6\sin\eta'\,e^{-\eta'} + e^{-2\eta} + 4\eta' - 9 + O(\sigma^{-\frac{1}{2}})\}, \tag{3.28}$$

and the corresponding stream function in the outer layer is given by

$$\psi_s^o = \frac{R'}{2\sigma^3 R^7}\{\eta^4 - \eta^2 R^2\}\{1 + O(\sigma^{-\frac{1}{2}})\} - \frac{9R'}{2^{\frac{3}{2}}\sigma^{\frac{7}{2}}R^8}\{\eta^4 - 2\eta^2 R^2\} + O(\sigma^{-4}). \tag{3.29}$$

Thus ψ_s^o has been evaluated explicitly only up to order σ^{-3}. However, we have shown the order-$\sigma^{-\frac{7}{2}}$ term proportional to $\eta^4 - 2\eta^2R^2$ explicitly since it is required in order that (3.28) and (3.29) match at the edge of the Stokes layer. The term of order $\sigma^{-\frac{7}{2}}$ proportional to $\eta^4 - \eta^2 R^2$ will clearly be of order σ^{-4} at the edge of the Stokes layer and so is not required explicitly for matching up to order $\sigma^{-\frac{7}{2}}$.

The flux through the pipe associated with (3.28) and (3.29) is clearly zero to order $\sigma^{-\frac{3}{2}}$ since $\psi_s^o|_{\eta=0} - \psi_s^i|_{\eta'=0}$ is $O(\sigma^{-4})$. We recall that $Q = 0$ corresponds to the ends of the pipe having the same mean radius. Alternatively if the ends of the pipe do not have the same mean radius Q can be made as small as we please by letting K tend to infinity in (3.22). The dominant steady streaming is then given by (3.28) and (3.29). The steady streaming in the high frequency limit when ϵ is not zero is evaluated in the appendix.

4. Calculation of the steady streaming for small σ

When σ is small the Stokes layer completely fills the pipe and there is no need to split the flow field into separate regions. We again solve for the order-R_M axisymmetric steady streaming velocity $(g_s, 0, f_s)$. We first consider the form of the Stokes flow for small σ. If we expand the Bessel functions appearing in (2.23) and (2.24) using the series form for Bessel functions of small argument we obtain

$$p'_{00} = (-i\alpha/R^4)\{\gamma_0 + \tfrac{1}{6}\gamma_0 i\sigma R^2 + \gamma_1 i\sigma + O(\sigma^2)\},$$

where

$$\gamma_0 = \left\{\int_0^K \frac{d\zeta}{R^4}\right\}^{-1}, \quad \gamma_1 = -\frac{\gamma_0^2}{6}\int_0^K \frac{d\zeta}{R^2},$$

and for convenience we choose $\alpha = \gamma_0^{-1}$. This is again equivalent to defining the typical axial velocity U_0 in terms of the amplitude of the applied pressure difference. We can then write p'_{00} in the form

$$p'_{00} = -\frac{i}{R^4}\left\{1 + \frac{i\sigma R^2}{6} + \frac{\gamma_1}{\gamma_0} i\sigma + O(\sigma^2)\right\} \tag{4.1}$$

and use (2.21) to show that

$$\begin{aligned}
f_{00} &\sim -i\{(\eta^2 - R^2)/4R^4\} + O(\sigma),\\
g_{00} &\sim -i\eta\{(\eta^2 - R^2)/4R^5\}R' + O(\sigma),\\
f_{01} &\sim (iS/2R^3)(\eta/R)^M \cos M\theta + O(\sigma),\\
g_{01} &\sim \frac{iSR'}{2R^3}\left(\frac{\eta}{R}\right)^{M+1}\cos M\theta + \frac{i}{4}\left\{\left(\frac{\eta}{R}\right)^{M+1} - \left(\frac{\eta}{R}\right)^{M-1}\right\}\left\{\frac{SR'}{R^3} - \frac{S'}{R^2}\right\}\cos M\theta + O(\sigma),\\
h_{01} &\sim \frac{i}{4}\left\{\left(\frac{\eta}{R}\right)^{M+1} - \left(\frac{\eta}{R}\right)^{M-1}\right\}\left\{\frac{S'}{R^2} - \frac{SR'}{R^3}\right\}\sin M\theta + O(\sigma).
\end{aligned}$$

Using the notation of §3 we can see that g_s, f_s and p_s are determined by (3.10) together with the boundary conditions

$$f_s = g_s = 0 \quad \text{at} \quad \eta = R, \tag{4.2a, b}$$

$$p_s(K) - p_s(0) = 0. \tag{4.2c}$$

We also require that g_s, f_s and p_s be regular at $\eta = 0$. If we now let σ tend to zero in (3.10) and use (2.21a), (2.25a), (4.1) and the series expansion of Bessel functions we can show that

$$\left\{\frac{\partial^2}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\right\}f_s = p'_s + \frac{R'}{16R^9}\{-\eta^4 + 2\eta^2R^2 - R^4\} + O(\sigma), \tag{4.3a}$$

$$\partial(\eta g_s)/\partial\eta + \eta\,\partial f_s/\partial\zeta = 0, \tag{4.3b}$$

and the solution of $(4.3a)$ which is regular at $\eta = 0$ and satisfies $(4.2a)$ is

$$f_s = p_s'\left\{\frac{\eta^2-R^2}{4}\right\} - \frac{R'}{1152R^9}\{2\eta^6 - 9\eta^4R^2 + 18\eta^2R^4 - 11R^6\} + O(\sigma). \quad (4.4a)$$

We can now substitute for f_s into $(4.3b)$ and integrate from 0 to η to obtain

$$\eta g_s = -\frac{p_s''}{16}\{\eta^4 - 2\eta^2R^2\} + \frac{p_s'}{4}R'R\eta^2 + \left(\frac{R'}{4608R^9}\right)'\{\eta^8 - 6\eta^6R^2 + 18\eta^4R^4 - 22\eta^2R^6\}$$
$$+\frac{R'^2}{384R^8}\{-\eta^6 + 6\eta^4R^2 - 11\eta^2R^4\} + O(\sigma), \quad (4.4b)$$

where we have used the fact that g_s is regular at $\eta = 0$ to show that ηg_s is zero there. If we put $\eta = R$ in $(4.4b)$ and use $(4.2b)$ we obtain the Reynolds equation for the pressure, which we integrate once to get

$$16C/R^4 = p_s' - R'/32R^5 + O(\sigma). \quad (4.4c)$$

C is a constant which, after integrating both sides of the above equation from $\zeta = 0$ to $\zeta = K$ and using $(4.2c)$, is found to be given by

$$C = 2^{-11}\{R^{-4}(K) - R^{-4}(0)\}\bigg/\int_0^K \frac{d\zeta}{R^4} + O(\sigma) \quad (4.5)$$

and C is therefore of order σ if the ends of the pipe are of the same mean radius. It can easily be shown that all higher-order terms in (4.5) also vanish in this case. If we introduce the stream function ψ_s defined by (3.24) we can show that

$$\psi_s = C\left\{\frac{\eta^4 - 2\eta^2R^2}{4R^4}\right\} - \frac{R'}{4608R}\left\{\left(\frac{\eta}{R}\right)^8 - 6\left(\frac{\eta}{R}\right)^6 + 9\left(\frac{\eta}{R}\right)^4 - 4\left(\frac{\eta}{R}\right)^2\right\} + O(\sigma). \quad (4.6)$$

Thus we see that if $R(K) = R(O)$ there is no net flux through the pipe. We recall that this was also the case in §3 when $R(K) = R(O)$. The steady streaming in the low frequency limit when ϵ is not equal to zero is discussed in the appendix.

5. Discussion of results

We have seen that in both the low and the high frequency limits the geometry of the pipe is crucial in determining the nature of the induced steady streaming. In particular the difference between the mean radii of its ends plays an important role. If this difference is zero then the steady streaming consists of regions of recirculation confined between the nodes of the pipe (i.e. where R' is zero). If the ends of the pipe do not have the same mean radius then there is a component of velocity which represents a steady flow towards the wider end of the pipe. In the high frequency limit the latter component of velocity in fact dominates the steady streaming whereas in the low frequency limit the two types of steady streaming appear at the same order of magnitude in σ. In order to see why this is so we return to $(3.11a)$.

We notice that the right-hand side of this equation contains two types of terms produced by the Reynolds stresses associated with the basic oscillatory flow. Clearly the term incorporated in Φ, namely $(2\sigma^3)^{-1}\{\tilde{p}_{00}'p_{00}'' + p_{00}'\tilde{p}_{00}''\}$, is uniform across the pipe whereas the term proportional to $e^{-\eta'}$ is exponentially small away from the Stokes layer. We can show by replacing the former term by its asymptotic form for large σ that both terms are of order σ^{-3}. However the former term is clearly of order σ^{-2} when the corresponding equation in the outer region is considered, the extra σ^{-1} factor in (3.11 *a*) appearing when the differential operator $\partial^2/\partial\eta^2+\eta^{-1}\partial/\partial\eta$ is written in terms of η'. The pressure gradient p'_s clearly depends on both terms. If we substitute for Φ from (3.12) into (3.21) we can show that

$$p'_s-\frac{16Q}{R^4} = -\frac{1}{4\sigma^2}\{p'_{00}\tilde{p}''_{00}+\tilde{p}'_{00}p''_{00}\}+\frac{R'}{2\sigma^3R^3}\left[1-\frac{9}{(2\sigma)^{\frac{1}{2}}R}\right]+O(\sigma^{-\frac{7}{2}}). \quad (5.1)$$

Hence the Reynolds-stress term independent of η' in (3.11 *a*) is balanced by an identical term in the pressure gradient. The other term on the right-hand side of (5.1) is of order σ^{-3} and arises from the terms proportional to $e^{-\eta'}$ in (3.11 *a*). Since the first term on the right-hand side of (5.1) is of order σ^{-2} we see that the two types of Reynolds-stress term in (3.11 *a*) affect p'_s at different orders in σ.

The constant Q is determined by there being no net pressure difference between the ends of the pipe arising from the terms on the right-hand side of (5.1). If the ends of the pipe have the same mean radius then Q is identically zero. Otherwise we recall that Q is of order σ^{-2} and the velocity field is dominated by the terms proportional to Q. The steady streaming is then given by

$$(g_s, 0, f_s) = \frac{\{R^{-4}(K)-R^{-4}(0)\}}{16\displaystyle\int_0^K \frac{d\zeta}{R^4}}\frac{\{\eta^2-R^2\}}{\sigma^2R^4}\left(\frac{R'\eta}{R}, 0, 1\right)+O(\sigma^{-\frac{5}{2}}) \quad (5.2)$$

and we can easily show that the stream surfaces associated with (5.2) are given by

$$\eta = \lambda R, \quad 0 \leqslant \lambda \leqslant 1. \quad (5.3)$$

A related effect has been reported by Ünlüata & Mei (1970), who considered mass transport in water waves. In the case of a wave tank closed at the far end, along which there is no net flux, they found that there was an induced steady pressure gradient. Similarly there is an induced steady pressure gradient associated with (3.28) and (3.29) in the high frequency limit and with (4.6) with C equal to zero in the low frequency limit. However, in our problem this steady pressure gradient is balanced by an equal and opposite one associated with the velocity component proportional to that given by (5.1).

In figure 1 we show the steady streaming in a wavy axisymmetric pipe whose ends have the same mean radius. In figure 2 we show the steady streaming in the Stokes layer at the pipe wall in more detail. The steady streaming shown in figure 2 is qualitatively similar to that found by Lyne (1971 *b*), who considered oscillatory viscous flow adjacent to a wavy wall. Our results correspond to the wavelength of the wall being much greater than both the thickness of the Stokes layer at the wall and the amplitude of oscillation of a fluid particle far from the wall. Moreover, Hall (1973) has considered oscillatory viscous flow in a two-dimensional channel of slowly varying depth. If one of the walls of the

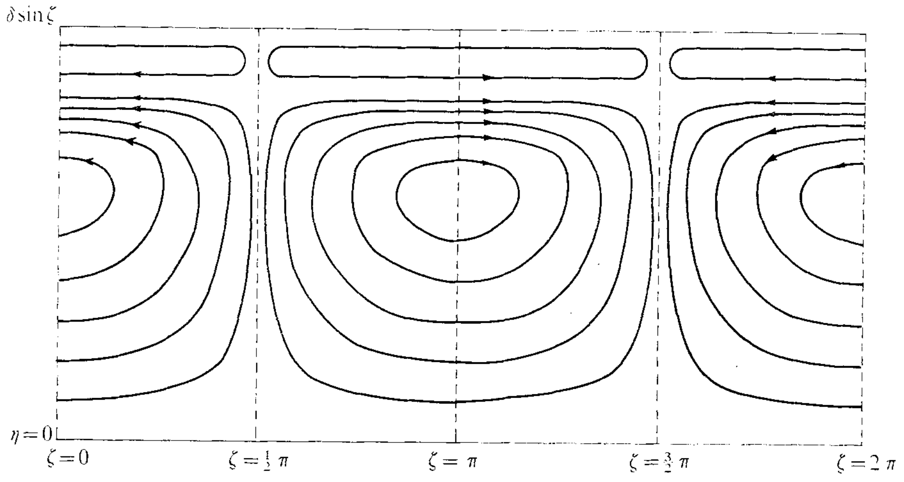

FIGURE 1. Steady streaming in a pipe defined by $\eta = 1-\delta\sin\zeta$, $0 \leqslant \zeta \leqslant 2\pi$, with $\sigma^{\frac{1}{2}}\delta \ll 1$ and $\sigma \gg 1$.

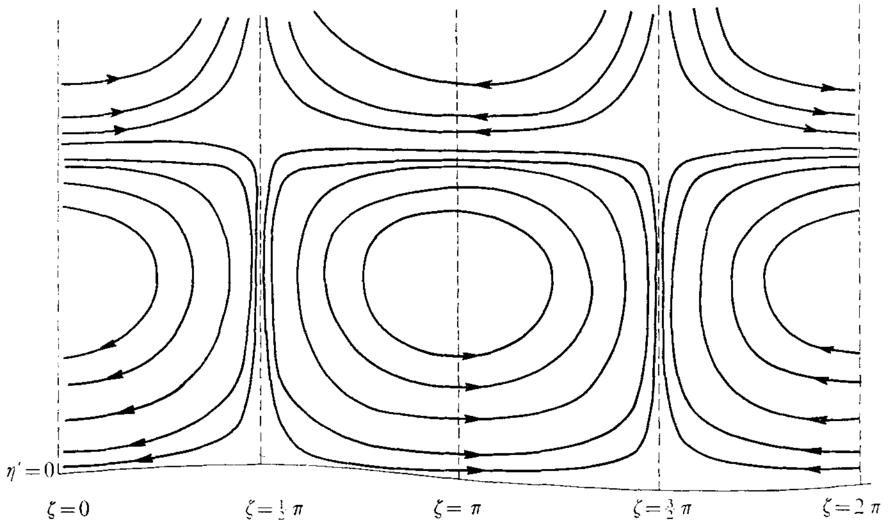

FIGURE 2. Steady streaming in the Stokes layer in a pipe defined by $\eta = 1-\delta\sin\zeta$, $0 \leqslant \zeta \leqslant 2\pi$, with $\sigma^{\frac{1}{2}}\delta \ll 1$ and $\sigma \gg 1$.

channel is taken to be wavy then the steady streaming in the Stokes layer at the wall is found to be identical to that found by Lyne.

In figure 3 we have sketched the steady streaming given by (4.6) for a pipe defined by

$$\eta = 1-\tfrac{1}{2}\exp-(\zeta-4)^2, \quad 0 \leqslant \zeta \leqslant 8.$$

The ends of this pipe have the same radius and so C in (4.6) is zero. In contrast to the high frequency limit we see that there is no region of recirculation near the pipe wall. The flow is such a pipe might be of some interest as a model for oscillatory flow in a narrow constricted blood vessel. However, in such a flow the condition that the pipe radius changes slowly would be violated and so δ, defined by (1.2), would not be small.

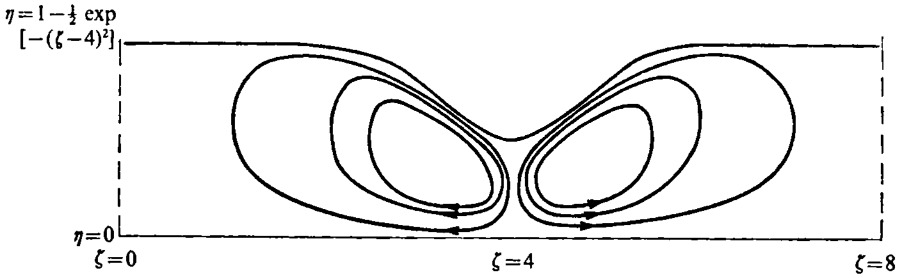

FIGURE 3. Steady streaming in a pipe defined by $\eta = 1-\frac{1}{2}\exp[-(\zeta-4)^2]$, $0 \leqslant \zeta \leqslant 8$, with $\sigma \ll 1$.

Finally, we compare the order of magnitude of the high frequency steady streaming given by (3.28) with that found by Lyne (1971*a*) for oscillatory flow in a curved pipe. A calculation shows that in the Stokes layers of these flows the ratio of typical axial steady velocities for flows with basic velocities of similar order and pipes of similar radius is L/R_0, where R_0 is the radius of curvature of the curved pipe. Thus we might expect that for flow in a curved pipe of varying radius the effects of curvature and narrowing are equally important as far as the Stokes-layer type of steady streaming is concerned. The steady streaming of the form given by (5.2) would clearly be more important than both the latter contributions since, as shown by (3.23) and (3.26), this effect appears at lower order in σ. It should also be pointed out that the steady streaming evaluated by Lyne had no component along the pipe.

Let us consider a pipe whose cross-section varies monotonically. The steady streaming in the high frequency limit will then be dominated by that given by (5.2) and will represent a steady flow towards the wider end of the pipe. The streamlines for such a flow will then be as given by (5.3). Lyne (1971*a*) discussed the relevance of his work to the flow in curved part of the human aorta. The parameters δ and σ for such a flow are typically of order 0·001 and 10·0 respectively whilst R_M is of order 100·0. Thus our theory is not strictly applicable but it is likely that the effect of narrowing of the aorta is at least as important as the effect of curvature as far as the steady streaming is concerned.

In the appendix we see that the work of §§3 and 4 can be extended to evaluate the steady streaming in a pipe which is slightly non-axisymmetric. The results of the high frequency order-ϵR_M calculation show that if $R(K) = R(O)$ then the steady streaming is confined to the Stokes layer and has no azimuthal component. If $R(K) \neq R(O)$ the steady streaming persists throughout the pipe and is found to have no azimuthal component of velocity if

$$S = \gamma R,$$

where γ is a constant. This corresponds to a pipe of uniform cross-sectional shape. Full details of the non-axisymmetric steady streaming can be found in the author's thesis (Hall 1973).

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Appendix

We now determine the order-ϵ corrections to the axisymmetric steady streaming which we evaluated in §§3 and 4. Suppose that we denote the steady parts of G_{11}, H_{11}, F_{11} and P_{11} by G_s, H_s, F_s and P_s respectively, then we can use (2.5), (2.9), (2.10) and (2.19) to show that

$$\partial P_s/\partial \eta = \partial P_s/\partial \theta = 0, \tag{A1a, b}$$

$$\nabla^2 F_s = P'_s + \frac{1}{4}\left\{f_{00}\frac{\partial \tilde{f}_{01}}{\partial \zeta} + f_{01}\frac{\partial \tilde{f}_{00}}{\partial \zeta} + g_{00}\frac{\partial \tilde{f}_{01}}{\partial \eta} + g_{01}\frac{\partial \tilde{f}_{00}}{\partial \eta} + \frac{h_{00}}{\eta}\frac{\partial \tilde{f}_{01}}{\partial \theta} + \frac{h_{01}}{\eta}\frac{\partial \tilde{f}_{00}}{\partial \theta}\right\} + \text{c.c.}, \tag{A2a}$$

$$\frac{\partial}{\partial \eta}(\eta G_s) + \frac{\partial H_s}{\partial \theta} + \eta \frac{\partial F_s}{\partial \zeta} = 0, \tag{A2b}$$

$$f_s + \epsilon F_s = g_s + \epsilon G_s = \epsilon H_s = O(\epsilon^2) \quad \text{at} \quad \eta = R + \epsilon S \cos M\theta, \tag{A3a, b, c}$$

where f_s and g_s are now assumed known. From (2.7) and (2.9) we can show that

$$P_s(K) - P_s(0) = 0, \tag{A4}$$

and if we eliminate p^+ from (2.5a, b), substitute for g, h, and f from (2.9) into the resulting equation and equate terms of order $\epsilon R_M \delta$ we obtain

$$\begin{aligned}
&\left\{\frac{\partial^2}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\eta^2}\right\}\left[\frac{\partial}{\partial \eta}(\eta H_s) - \frac{\partial G_s}{\partial \theta}\right] \\
&\quad = \frac{1}{4}\left[\frac{\partial}{\partial \eta}\left(\eta g_{00}\frac{\partial \tilde{h}_{01}}{\partial \eta} + \eta g_{01}\frac{\partial \tilde{h}_{00}}{\partial \eta} + \eta f_{00}\frac{\partial \tilde{h}_{01}}{\partial \zeta} + \eta f_{01}\frac{\partial \tilde{h}_{00}}{\partial \zeta} + h_{00}\frac{\partial \tilde{h}_{01}}{\partial \theta}\right.\right. \\
&\qquad\qquad \left. + h_{01}\frac{\partial \tilde{h}_{00}}{\partial \theta} + h_{00}\tilde{g}_{01} + h_{01}\tilde{g}_{00}\right) \\
&\qquad \left. - \frac{\partial}{\partial \theta}\left(g_{00}\frac{\partial \tilde{g}_{01}}{\partial \eta} + g_{01}\frac{\partial \tilde{g}_{00}}{\partial \eta} + f_{00}\frac{\partial \tilde{g}_{01}}{\partial \zeta} + f_{01}\frac{\partial \tilde{g}_{00}}{\partial \zeta} + \frac{h_{00}}{\eta}\frac{\partial \tilde{g}_{01}}{\partial \theta} + \frac{h_{01}}{\eta}\frac{\partial \tilde{g}_{00}}{\partial \theta}\right)\right] + \text{c.c.}
\end{aligned} \tag{A 5}$$

Together with the conditions that F_s and P_s are independent of θ at $\eta = 0$ and G_s and H_s vary like $\cos\theta$ and $\sin\theta$ there (A2)–(A5) completely specify the problem for G_s, H_s, F_s and P_s. The solution of this system is long and tedious and we briefly summarize the method here. The full details are given by Hall (1973).

The complexity of the right-hand sides of (A2a) and (A5) leads us to consider the Stokes and outer layers separately again in the high frequency limit. We use (G_s^i, H_s^i, F_s^i) and (G_s^o, H_s^o, F_s^o) to represent (G_s, H_s, F_s) in the Stokes and outer layers respectively. Equations (A2a) and (A5) are written in appropriate forms for these layers and solved such that their solutions in different layers match where the layers meet. If the values of F_s^i and F_s^o are then substituted into the equation of continuity in these layers we obtain equations for G_s^i, H_s^i, G_s^o and H_s^o. These can be solved using the solution of (A5) in each layer to give G_s^i, H_s^i, G_s^o and H_s^o. If we then apply the conditions that (G_s^i, H_s^i) and (G_s^o, H_s^o) match at the edge of the Stokes layer together with (A3) and (A4), all the unknown functions of ζ appearing in F_s^i, F_s^o, G_s^i, etc., can be determined. It is again found that the solution is greatly influenced by whether or not Q, defined by (3.21), is zero.

If Q is not zero it is not necessary to distinguish between the two layers and we can write

$$G_s = \frac{A_0}{o^2}\left[\left\{\left(\frac{\eta}{R}\right)^{M+1} - \left(\frac{\eta}{R}\right)^{M-1}\right\}\left\{\frac{S'}{R^2}-\frac{SR'}{R^3}\right\} - 2\left(\frac{\eta}{R}\right)^{M+1}\frac{SR'}{R^3}\right]\cos M\theta + O(\sigma^{-\frac{5}{2}}), \tag{A6a}$$

$$H_s = \frac{A_0}{\sigma^2}\left[\left\{\left(\frac{\eta}{R}\right)^{M+1} - \left(\frac{\eta}{R}\right)^{M-1}\right\}\left\{\frac{SR'}{R^3}-\frac{S'}{R^2}\right\}\right]\sin M\theta + O(\sigma^{-\frac{5}{2}}), \tag{A6b}$$

$$F_s = (-2A_0 S/\sigma^2 R^3)/(\eta/R)^M \cos M\theta + O(\sigma^{-\frac{5}{2}}), \tag{A6c}$$

where A_0 is given by (3.23*c*). However, when Q is zero we must distinguish between the layers and we have

$$G_s^i = (SR'^2/2^{\frac{1}{2}}\sigma^{\frac{5}{2}}R^5)\{e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\}\cos M\theta + O(\sigma^{-3}), \tag{A7a}$$

$$H_s^i = O(\sigma^{-3}), \tag{A7b}$$

$$F_s^i = (SR'/2^{\frac{1}{2}}\sigma^{\frac{5}{2}}R^5)\{e^{-2\eta'} + 3\sin\eta' e^{-\eta'} - 2\cos\eta' e^{-\eta'} - \eta'\cos\eta' e^{-\eta'}\}\cos M\theta + O(\sigma^{-3}), \tag{A7c}$$

$$G_s^o = H_s^o = F_s^o = O(\sigma^{-3}), \tag{A7d, e, f}$$

and so in this case the dominant steady streaming of order ϵR_M is confined to the Stokes layer and has no swirling component. Similarly if we choose $S = \gamma R$, where γ is a constant, the dominant steady streaming of order ϵR_M, given by (3.35), will have no such component. This particular simplification corresponds to flow in a pipe of constant cross-sectional shape.

In the low frequency limit a similar approach can be used but there is no need to split the flow into separate regions. The results of performing such a calculation are particularly complicated even in the special case $S = \gamma R$ and are not particularly interesting. Details of this solution are given in the author's thesis (Hall 1973).

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